$$\text{weighted arithmetic mean} = \frac{p'_1q_1+p'_2q_2+\cdots\cdots\cdots+p'_nq_n}{q_1+q_2+\cdots\cdots\cdots q_n},$$

and the

$$\text{weighted harmonic mean} = \frac{p'_1q_1+p'_2q_2+\cdots\cdots\cdots+p'_nq_n}{\frac{1}{p'_1}p'_1q_1+\frac{1}{p'_2}p'_2q_2+\cdots\cdots+\frac{1}{p'_n}p'_nq_n}$$

$$=\frac{p'_1q_1+p'_2q_2+\cdots\cdots\cdots+p'_nq_n}{q_1+q_2+\cdots\cdots\cdots q_n}.$$

The weighted arithmetic and harmonic means, with properly chosen weights, therefore, are identical and neither can be called "biased" with respect to the other. When Professor Fisher uses the word "bias," it is not clear to me "with respect to what" the average is biased.

REJOINDER BY IRVING FISHER

First let me thank my critics for their kind words and express my gratification that they agree with me so far as they do. The disagreements which have been expressed are, I believe, in many cases more apparent than real; and I venture to hope that any real disagreements remaining may melt away as the field is threshed over a little more thoroughly.

I am very glad to join heartily with Mr. Walsh in advocating the practical use of what I call the "ideal" formula whenever the necessary data as to quantities for both years are available. It is only when these data are not available—when merely the quantities for the base year are available, that I should propose some other formula as a makeshift.

But Professor Persons and Professor Mitchell are unwilling, at present, to agree with Mr. Walsh and myself on the "ideal" formula as the best for all purposes.

As I understand it, they express doubt rather than disbelief, and make the appeal that the particular *purpose* for which an index number is to be used must surely make a difference.

I quite agree that the purpose of an index number is a very important factor in determining what is the best index number. This is certainly true as to the elements of an index number other than the formula—the character and number of commodities, for instance. But as to the mathematical formula itself, I take a different view.

It is true in the broad field of science, that various types of mathematical averages are best suited for various widely separated purposes.

For instance, to get the average speed of an automobile we should take the *arithmetical* average of the readings of the speedometer taken at short equal intervals of time, such as every second; the harmonic or geometric mean would give false results. On the other hand, we should take the *harmonic* average if the readings of the speedometer are taken at short equal *distances*, such as at every telegraph pole; the arithmetic average as well as the geometric average would then give false results. Again, if we have a weighing balance, or scales, with unequal arms (so that the figure for the weight of an object is found to be different according as it is weighed in one pan with the counterpoise in the other, or vice versa), the true result is obtained by taking the *geometric* mean of the two conflicting results; the arithmetic or the harmonic mean would then give wrong results. Still again, in my *Appreciation and Interest* I have shown that the average rate of interest of a series of successive rates of interest (such as used, for instance, in connection with discounting the proposed German indemnity) is a certain complicated and unsymmetrical average quite different from any suitable for any other purpose, so far as I know.

As to an index number, I would hold that an index number is itself a purpose. It is a purpose sufficiently different from the purposes in the other realms just cited, and sufficiently homogeneous within its own realm, to require certain definite general criteria of its own, whatever the sub-purpose within the domain of index numbers may be.

One could agree to the statement that one system of bookkeeping is good for one kind of business and another for another, without being thereby committed to the proposition that in some businesses one mathematical process should be used and in another, another—that in some the assets should be added together and in others, multiplied. Or, again, one could agree to the statement that one kind of scales is good for one kind of business, and another for another, without being thereby committed to the proposition that in correcting the two readings obtained by weighing in the two pans, the geometric mean is appropriate when we are weighing tea, but some other average must be used for sugar.

I should like to call attention to the fact that neither Professor Mitchell nor Professor Persons has pointed out a single specific case in which the sub-purpose would require that either of the two tests which I have indicated as the supreme tests should be disregarded.

Professor Persons offers, it is true, what would seem at first glance to be a crucial test of my formula for one purpose. He wishes to compute the correlation between an index number of prices and an index number of quantities, for twelve crops; and alleges that my pro-

posed "ideal" formula could not properly be used for this particular purpose. He thinks the correlation would be spoiled by the fact that the index number of prices involves quantities and the index number of quantities involves prices, thus mixing at the outset the two variables whose correlations he is seeking.

A little consideration will, I think, convince Professor Persons that the alleged fault in the formula is imaginary. Suppose that each one of his twelve commodities is such that its price varies inversely with its quantity or crop so that there exists a 100 per cent inverse correlation between the price of the commodity and the quantity—a perfect correlation. Were Professor Persons' misgivings well-founded, we ought to find that the index number of the twelve prices, weighted by the quantities according to the "ideal" formula, would fail to show a 100 per cent inverse correlation with the corresponding index number of quantities. But the fact is that it does show the full 100 per cent required! I shall not take the space to detail the proof. Professor Persons is well able to do this for himself. The case he cites merely adds another to the list of virtues of the "ideal" index number.

I take the liberty to hereby issue a friendly challenge to Professor Persons and Professor Mitchell to demonstrate some particular purpose for which it is desirable that an index number should not work both ways in time, or should not work both ways as to the p's and q's.

These two tests stare us in the face whenever we meet a formula for a weighted index number. For every weighted index number contains symbols for both prices and quantities, and symbols for both the given year and the base year. My tests merely require that, in both cases, the pair of symbols should be treated alike.

The formula which I call the "ideal" is, it is true, not the only possible formula which will conform to both these tests. But all the formulae which do conform to these two tests will yield figures which differ by only a small fraction of 1 per cent.

I believe that we shall reach substantial agreement, from a practical point of view at least, if we can agree merely to discard formulae which fail to conform, even fairly well, to the year-reversal test. We shall thereby rule out all the weighted arithmetic and harmonic averages (except those which reduce to price aggregates) and all geometric averages which do not have symmetrical or "double" weighting. The gross errors in the actual use of index numbers come from disregarding this year-reversal test.

In this connection, I should like to point out what seems to be an error in Mr. Walsh's statement. The "double-weighting" which he stresses so much is valuable for the geometric averages, but it is not

always the best weighting. If, to give a specific example, we are using an *arithmetic* average for an index number of *prices*, the best weighting will be one in which the *prices* of the base year are used. The downward bias introduced by using the prices of the base year will offset the natural upward bias inherent in the arithmetic average. A bias-free weight is as out-of-place with an arithmetic or harmonic average as a biased weight is for a geometric average.

As to Westergaard's test, on which Mr. Walsh lays so much stress and which is evidently considered important by Professor Persons (the test, that is, by which a direct comparison between the base year and any given year is required to yield the same result as an indirect comparison found by multiplying the index number of some third year in terms of the base year by the index number of the given year in terms of the third year), I have come to three conclusions: first, a *complete* fulfilment of that test by a formula for a weighted index number is impossible; second, it is not desirable; and third, the "ideal" index number comes closer to fulfilling this test than any other.

It follows that Professor Persons is demanding the impossible when he asks that an index number on a fixed base should be usable for exact comparisons between two given years. Furthermore, his demand is more nearly satisfied by my proposed "ideal" index number than by any other, and considerably more than by the weighted geometric mean mentioned favorably by him and which Professor Day has employed for his index number of quantities.

For all practical purposes, the "ideal" index number does fulfil Professor Persons' requirement, as the deviation is usually less than half of 1 per cent even under the conditions of wide dispersement of prices and quantities in the war period of 1914–18.

The only forms of index number, out of several hundred compared, which *absolutely* meet the Westergaard test (*i. e.*, are not subject to "the limitations to dual comparisons" complained of by Professor Persons) are the *simple* geometric, median, mode and price-sum! No one has claimed, or could claim, that these are superior to weighted averages.

Professor Persons speaks of the simple geometric mean as lying between the simple arithmetic and harmonic means, and then remarks that "the weighted means, however, are not always arranged in this order." This statement is erroneous if it is assumed that the *same weights* are used in the three means compared. In the instance cited by Professor Persons, where a weighted arithmetical average is identical with a weighted harmonic, the weights are different in the two cases. This instance is like that cited in my *Purchasing Power of Money*

(formulae 11 and 12, see pages 395 and 397 and headings 11 and 12 of table opposite page 418). In this and other like cases the bias of the arithmetic or harmonic average is counterbalanced by the opposite bias of the weighting used.

Where quantities for the given years are not available, so that what Professor Persons calls "variable" weights cannot be used, an index number very close to the ideal and free from any gross bias can be used, namely, the above mentioned arithmetical average, the weights being the values of the base year, which is identical with the harmonic average, the weights being the products of the quantities of the base year multiplied by the prices of the given year, which is identical also with the price-aggregate index number. This is superior to the geometric average with base-value weights favored, apparently, by Professor Persons. This geometric average is inferior just because it has no bias to neutralize the downward bias of the base-value weighting (associating, as that weighting does, high prices with low weights, and low prices with high weights).

In further reference to Mr. Walsh's criticism, I would say that there is much in my full paper which was not given at the Atlantic City meeting, partly because of lack of time and partly because the calculations have not all been finished. The calculations include those for Westergaard's test and formulae in which the mixture of the base year and given year is accomplished *in the weights* and not by taking the geometric mean of the completed formulae. I had no thought of using the former method "exclusively." I think the complete paper will also show that the work of calculating all four, instead of only two, of the weighted averages was not wasted.

Mr. Walsh is right in noting a slip which I made in the first abstract sent to those who were to discuss my paper at Atlantic City. But the slip was not quite what he took it to be. There are, of course, very many index numbers which fulfil *one* of the two supreme tests. I was speaking of formulae which fulfil *both* tests. What I should have said is that there are only a few such formulae and that among them the "ideal" is distinguished by its simplicity and by the fact that the two components of which it is formed are very close together to start with, as well as by other virtues, all of which may be fairly said to make it unique.

I hope I have, by this time, made it clear "with respect to what" an index number may be called "biased." It is with respect to its own assumed standard. If Professor Persons will collect data for the varying prices and quantities in any two towns of the United States, such as Boston and New York, he will find that any among what

I have called "upward biased" index numbers will make Boston appear to have a higher price level than New York (if New York prices are assumed as the base) and yet make New York appear to have a higher price level than Boston (if Boston prices are assumed as the base); while, reversely, the downward biased index numbers will make each city lower priced than the other! Is it possible that an index number can be good for *any* purpose whatever if it is so consistently inconsistent as to make every town higher priced than every other? Such an index number stands self-condemned!